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Convective derivatives and Reynolds transport in curvilinear time-dependent coordinates

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Abstract
A fully covariant formulation of kinematics and dynamics of fluid flows and heat transfer is developed in time-dependent curvilinear coordinate systems. These moving and deformable reference frames have the same properties of a fluid motion and they can be successfully applied to a variety of problems ranging from numerical analysis to theoretical physics. The classical Reynolds transport theorem, the Euler formula and the acceleration addition theorem are extended to these general types of coordinates through a generalization of the convected differentiation concept originally introduced by Oldroyd (1950 *Proc. R. Soc.* A **200** 523–41). A rigorous formulation of dynamical equations and conservation laws in curvilinear time-dependent coordinates could be the key for the construction of variational principles based on the method of constrained variations.

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1. Introduction

This paper concerns the tensorial formulation of kinematics and dynamics of fluid flows and heat transfer in curvilinear time-dependent coordinate systems. The tensorial formulation of the basic equations in fluid mechanics is obviously a classical topic (e.g. Aris (1989), Batchelor (1967)). However, the work in the literature has focused almost exclusively on fixed coordinate systems, and the extension to moving curvilinear coordinates is rarely discussed. The fundamental questions we consider are: how do we describe the physical motion of a fluid from another time evolving flow of coordinates? How do dynamical equations and conservation laws need to be modified? Does there exist a *privileged* coordinate flow having special properties for a given physical fluid flow or a field variable? Obviously, these questions are not new and researchers have worked on them for decades (e.g., Viviand (1974), Ogawa and Ishiguro (1987), Rosenfeld and Kwak (1991), Liseikin (1999)).

However, Luo and Bewley (2004) and Thiffeault (2001) have recently shown that an elegant answer can be rigorously formulated by using tensor calculus in curvilinear time-dependent coordinates. These interesting extensions led me to further proceed along this direction and reformulate classical results from kinematics such as the acceleration addition theorem and the Reynolds transport theorem under this new perspective.

Dynamical equations and conservation laws expressed in general time-dependent curvilinear coordinates have many interesting applications ranging from numerical analysis (Luo and Bewley (2004), Rosenfeld and Kwak (1991), Ogawa and Ishiguro (1987)) to theoretical physics. For instance, the numerical solution of time-dependent equations sometimes requires the application of moving grids and corresponding coordinate transformations, which are dependent of time. The introduction of these time-dependent transformations enables one to compute an unsteady solution on a fixed uniform grid by the numerical solution of the transformed set of equations. This can be done both *locally*, i.e. element-wise, or *globally*. From a theoretical point of view, recent developments on variational methods suggest that a generalized formulation in time-dependent coordinates could be the key for determining variational principles based on the method of constrained variations (e.g., Bretherton (1970), Larsson (2003), Holm *et al* (1998)).

There is in fact a very strong connection between formulating the equations of motion of a physical system in time-dependent curvilinear coordinates and the existence of a variational principle for such set of equations. This connection has been clearly pointed out a long time ago by Herivel (1955) in the context of perfect fluid flows. He recognized that a very simple variational principle for the Euler equations can be formulated if Lagrangian coordinates are considered. Historically, the extension of this principle to fixed coordinates has led to many technical difficulties (e.g. Drobot and Rybarski (1958), Serrin (1959), Eckart (1960), Seliger and Whitham (1968), Finlayson (1972), Mobbs (1982), Salmon (1988), Morrison (1998)), mostly due to the appearance of velocity potentials whose physical significance is somehow obscure. To overcome these problems Bretherton (1970) first proposed a hybrid Hamilton's principle for perfect fluids based on *constrained* variations (see, e.g., Larsson (1996, 2003), Wilhelm (1979) for recent developments and applications). The key idea is to compute a field variation induced by a small perturbation of particle paths in convected coordinates and transform it back to fixed coordinates[1]. As clearly seen, this method basically relies on a functional perturbation theory which is built upon time-dependent curvilinear coordinate systems. The fundamental connection between the form of the equations and the existence of a variational principle has also been recognized by Tonti (1984) (see also Tonti (1969a, 1969b), Vainberg (1964), Finlayson (1972), Filippov (1989)) under a different perspective.

In order to obtain the equations of fluid mechanics in curvilinear time-dependent coordinate systems, Luo and Bewley (2004) have recently proposed an extension of the Reynolds transport theorem which has been left without a rigorous proof. In this paper, we provide such a proof through a generalization of the convected differentiation concept originally introduced by Oldroyd (1950).

This paper is organized as follows. In section 2 we introduce basic facts regarding the kinematical description of fluid flows in time-dependent curvilinear coordinates. In section 3 we consider the concept of intrinsic differentiation along a curve and we extend the proof of Luo and Bewley (2004) to tensors up to order two. In section 4 we discuss the Oldroyd (1950) convected derivative with respect to time and obtain its generalized expression in time-dependent coordinates. Previously unobserved connections between convected and intrinsic

[1] Today the method of constrained variations has a place within abstract mathematical theory in terms of Lie algebras with representations (e.g. Holm *et al* (1998)).

derivatives are reported as well. Section 5 is devoted to the tensorial formulation of the acceleration addition theorem. In section 6 we consider the Reynolds transport theorem and we provide a new proof based on the concept of convected differentiation. Examples of application are reported in section 7, where the equations of mass and energy conservation as well as momentum transport are determined. We also include a brief appendix treating the temporal differentiation of the metric tensor determinant in convected coordinates. We will extensively use tensor calculus throughout the entire paper. For excellent treatments see Weinberg (1972), Aris (1989) and Lovelock and Rund (1989).

2. Kinematics of fluid motion

We begin this section by emphasizing the fundamental difference which occurs between a *coordinate system* and an *observer* as these two concepts are frequently wrongly interchanged in tensor analysis. Let us briefly describe where the misunderstanding is. Consider a uniform and stationary fluid flow along the x-axis of a fixed Cartesian and orthogonal coordinate system. Let v_x be the velocity component along x. Let us also consider another Cartesian coordinate system whose axes are parallel to the aforementioned reference frame and whose motion proceeds along x exactly with velocity v_x. We formulate the following question: what are the fluid-flow velocity components with respect to the moving system? The answer is: the velocity components are exactly the same in both coordinate systems. This can be easily seen by examining the tensorial transformation rule for the contravariant velocity field. The reader might be skeptical about this result and rightly suggest that in the moving reference frame the fluid is *observed* to be in a no-motion state. These kind of statements presuppose that the velocity of the moving system is somehow subtracted from the flow velocity and this is obtained if a change of observer is performed. This is equivalent to evaluating the flow velocity through a material differentiation along fluid element paths as observed from moving coordinates. In other words, the tensorial transformation rule for the velocity field and the material differentiation along fluid element trajectories relative to moving coordinates lead to different results. The whole apparatus of tensor calculus, however, deals with the transformation of tensors corresponding to coordinate transformations, and when such a transformation is performed there is no change of observer.

A classical example of time-dependent coordinates is a rigidly rotating Cartesian system, widely used in fluid mechanics for the study of geostrophic flows or flows between spinning coaxial cylinders. A generalization of the rigid reference frame concept naturally yields to time-dependent curvilinear coordinates whose motion in space resembles *in toto* a physical fluid flow. In formulating the general relativity theory Einstein (1966) called these types of coordinates 'molluscs of reference' to emphasize the fact that their metric relations depend on space as well on time. The fluid mechanicist is definitely familiar with a particular type of such a coordinate flow, i.e. the Lagrangian (*material* or *convected*) coordinates. In this case physical fluid flow and coordinate flow basically coincide.

2.1. Description of trajectories, velocity and acceleration

Fluid flow is an intuitive physical notion which is represented mathematically by a continuous transformation of three-dimensional Euclidean space into itself. In order to set up this transformation we consider a fluid particle labeled by $\boldsymbol{\xi}$ and represent its trajectory in fixed Cartesian coordinates as $\widehat{\boldsymbol{x}}(\boldsymbol{\xi}, t)$. The parameter '$t$' is identified with the time while we will refer to $\boldsymbol{\xi}$ as *material* coordinates. The expression $\widehat{\boldsymbol{x}}(\boldsymbol{\xi}, t)$ actually determines an ensemble of paths, one for each specific choice of $\boldsymbol{\xi}$.[2]

Now we consider a transformation from the Cartesian system $\boldsymbol{x}$ to another curvilinear system $\boldsymbol{\eta}$. We allow this mapping to be time dependent and we write it as $\widehat{\boldsymbol{\eta}}(\boldsymbol{x}, t)$. This transformation has exactly the same properties of the fluid motion $\widehat{\boldsymbol{x}}(\boldsymbol{\xi}, t)$, i.e. it can be basically considered as another flow of particles $\boldsymbol{\eta}$ whose motion in Cartesian coordinates is readily obtained through an inversion of $\widehat{\boldsymbol{\eta}}(\boldsymbol{x}, t)$. The next question is: how does this coordinate flow see the trajectories of the physical particles? To answer this question we simply bind the $\boldsymbol{x}$ dependence of $\widehat{\boldsymbol{\eta}}(\boldsymbol{x}, t)$ to $\widehat{\boldsymbol{x}}(\boldsymbol{\xi}, t)$, obtaining $\widehat{\boldsymbol{\eta}}(\widehat{\boldsymbol{x}}(\boldsymbol{\xi}, t), t)$.

Using the same notation adopted by Luo and Bewley (2004), the relative velocity field is easily obtained as *material* derivative (i.e. derivative with respect to time keeping $\boldsymbol{\xi}$ constant) of $\widehat{\boldsymbol{\eta}}(\widehat{\boldsymbol{x}}(\boldsymbol{\xi}, t), t)$, i.e.

$$\frac{\mathrm{d}\widehat{\eta}^j}{\mathrm{d}t} = \frac{\partial\widehat{\eta}^j}{\partial t} + \frac{\partial\widehat{\eta}^j}{\partial x^k}\frac{\mathrm{d}\widehat{x}^k}{\mathrm{d}t} = -U^j + u^j, \tag{2.1}$$

where U^j are the velocity components of the coordinate flow while u^j are the velocity components of the physical flow. Both U^j and u^j are tensorial components[3] relative to moving coordinates $\boldsymbol{\eta}$. Equation (2.1) basically states the Galilean velocity addition theorem: the velocity of a particle relative to a moving coordinate system equals its absolute velocity minus the velocity of the reference system.

An additional material derivative of the relative velocity $u_R^k := u^k - U^k$ gives the relative acceleration

$$\frac{\mathrm{d}^2\widehat{\eta}^k}{\mathrm{d}t^2} = \frac{\partial u_R^k}{\partial t} + \frac{\partial u_R^k}{\partial \eta^j}u_R^j. \tag{2.2}$$

It is evident that (2.2) in general is a non-tensorial quantity. To see this we simply substitute the partial derivative $\partial u_R^k/\partial\eta^j$ with its covariant representation (Γ_{nj}^k denotes the affine connection)

$$\frac{\partial u_R^k}{\partial \eta^j} = u_{R;j}^k - \Gamma_{nj}^k u_R^n, \tag{2.3}$$

to obtain

$$\frac{\mathrm{d}^2\widehat{\eta}^k}{\mathrm{d}t^2} = \frac{\partial u_R^k}{\partial t} + u_{R;j}^k u_R^j - \Gamma_{nj}^k u_R^j u_R^n. \tag{2.4}$$

Equation (2.4) ensures covariance of the relative acceleration only for a restricted class of coordinate flows having vanishing affine connection coefficients Γ_{nj}^k. Cartesian and orthogonal reference frames in rigid motion belong to this class. We postpone the formulation of the acceleration addition theorem in its fully covariant form to section 5 after the concepts of intrinsic as well as convected differentiation are introduced.

[2] As is well known, material coordinates have different interpretations. They can be seen as a curvilinear mesh which is somehow glued to the physical flow and evolves with it. Another common and widely used interpretation is that ξ are coordinates of the fluid elements at a certain reference time t_0. In the first case, attention is focused more on dynamical aspects of the flow while in the second case more attention is devoted to the methodology which assigns a label to a fluid element, specifically $\xi = \widehat{x}(\xi, t_0)$. It is assumed that distinct physical particles remain distinct throughout the entire motion, i.e. two particles cannot collapse in one particle and one particle cannot split into two particles. This is equivalent to assuming that the fluid matter cannot detach or compenetrate during its motion. It is well known that this requirement restricts the class of transformations $\widehat{x}(\xi, t)$ to ones possessing continuous derivatives up to the third order in all variables (except possibly at certain singular surfaces, curves or points).

[3] For an illuminating discussion on the fundamental difference between tensorial and physical components of vectors and tensors see Truesdell (1953) or Aris (1989).

3. Intrinsic differentiation

In this section we introduce the concept of *intrinsic* differentiation of tensors along particle paths in time-dependent curvilinear coordinate systems. Let us briefly clarify why we need to consider such an operator and what its meaning is. It is well known from tensor analysis that a simple material differentiation of a tensorial field along a particle path produces a certain number of terms which are responsible for a non-tensorial behavior of the derived quantity. What does this mean? Consider as an example two different curvilinear coordinate systems and a particle path conveniently expressed in both of them. If we compute a material derivative of the particle velocity in both coordinate systems and we transform these results into a third coordinate system we will get two different acceleration vectors, as if the trajectory of the particle was not unique. This happens because the material derivative of a vector in general is not a tensor.

The intrinsic derivative operator is constructed so as to guarantee covariance (i.e. the independence from the choice of the coordinate system) of a temporal differentiation along particle paths relatively to *all* possible coordinate systems. In classical tensor analysis covariance is usually required with respect to *fixed* curvilinear coordinates. Within this framework the intrinsic derivative of a contravariant vector A^i is computed as (e.g. Aris (1989))

$$\frac{\delta A^i}{\delta t} = \frac{\partial A^i}{\partial t} + A^i_{;j} u^j, \tag{3.1}$$

where $A^i_{;j}$ denotes a covariant differentiation while u^i are particle velocity components. In ordinary fluid mechanics, intrinsic derivatives in fixed curvilinear coordinates are also widely known as *convective* derivatives[4]. This is due to the so-called convective rate of change (Batchelor (1967), p 73), identified by the second term appearing in (3.1).

The intrinsic derivative of a contravariant vector field in time-dependent coordinate systems has recently been obtained by Luo and Bewley (2004) in the context of Navier–Stokes equations representation. In the following subsection we extend the theory to second order tensors.

3.1. Intrinsic derivative of second order tensors

Following Luo and Bewley (2004), we use the quotient rule (e.g. Aris (1989)) to establish a tensorial quantity which defines the temporal variation of second order tensors along particle paths in general time-dependent curvilinear coordinates. To this end we construct the scalar field

$$\mathfrak{C} = A^i_j B^j C_i, \tag{3.2}$$

[4] We remark that the term 'convective derivative' is often used to denote also another type of derivative, namely the Oldroyd derivative (Oldroyd (1950)), which is discussed in the subsequent section 4. This naming has been adopted by Thiffeault (2001) section 5.1, Aris (1989) p 185, Gatski and Lumley (1978) p 626, Billington and Tate (1981) p 61, Rappaz *et al* (2001) p 306, Gatski (2001) p 26. Other authors refer to the Oldroyd derivative as 'convected time derivative'. The latter definition seems to be more popular, and maybe indicates more precisely the physical meaning of such a derivative. Oldroyd himself uses the term 'convected differentiation with respect to time' (Oldroyd (1950), p 530). Therefore, in order to avoid confusion, we shall reserve the term 'convected derivative' to denote the Oldroyd derivative and we shall equivalently call (3.1) the intrinsic derivative in fixed curvilinear coordinates or, more simply, the convective derivative.

where B^j and C_i are parallel contravariant and covariant vectors, respectively. It is well known that the concept of parallelism for vectors defined along arbitrary paths in curvilinear coordinates leads to the following requirements:

$$\frac{\mathrm{d}B^j}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^k}{\partial\eta^m}\right)\frac{\partial\widehat{\eta}^j}{\partial x^k}B^m, \tag{3.3}$$

$$\frac{\mathrm{d}C_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^k}{\partial\eta^i}\right)\frac{\partial\widehat{\eta}^m}{\partial x^k}C_m. \tag{3.4}$$

We differentiate (3.2) with respect to time following the physical particle $\boldsymbol{\xi}$:

$$\frac{\mathrm{d}\mathfrak{E}}{\mathrm{d}t} = \frac{\mathrm{d}A^i_k}{\mathrm{d}t}C_iB^k + A^i_k\frac{\mathrm{d}C_i}{\mathrm{d}t}B^k + A^i_kC_i\frac{\mathrm{d}B^k}{\mathrm{d}t} \tag{3.5}$$

and take into account the parallelism relations (3.3) and (3.4) to obtain

$$\frac{\mathrm{d}\mathfrak{E}}{\mathrm{d}t} = \left[\frac{\partial A^i_k}{\partial t} + \frac{\partial A^i_k}{\partial\eta^m}\frac{\mathrm{d}\widehat{\eta}^m}{\mathrm{d}t}\right]C_iB^k + \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^j}{\partial\eta^i}\right)\frac{\partial\widehat{\eta}^m}{\partial x^j}A^i_kC_mB^k}_{m\leftrightarrow i}$$

$$- \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^j}{\partial\eta^m}\right)\frac{\partial\widehat{\eta}^k}{\partial x^j}A^i_kC_iB^m}_{m\leftrightarrow k}.$$

An interchange of indices as illustrated above yields

$$\frac{\mathrm{d}\mathfrak{E}}{\mathrm{d}t} = \left[\frac{\partial A^i_k}{\partial t} + \frac{\partial A^i_k}{\partial\eta^m}\frac{\mathrm{d}\widehat{\eta}^m}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^j}{\partial\eta^m}\right)\frac{\partial\widehat{\eta}^i}{\partial x^j}A^m_k - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^j}{\partial\eta^k}\right)\frac{\partial\widehat{\eta}^m}{\partial x^j}A^i_m\right]C_iB^k. \tag{3.6}$$

The quotient rule tells us that the quantity within brackets must be a second order mixed tensor, which we define to be $\delta A^i_k/\delta t$ and call the *intrinsic* derivative of A^i_k along the trajectory of the physical particle $\boldsymbol{\xi}$ in coordinates $\boldsymbol{\eta}$. Now we express such a tensor in a manifestly covariant form. To this end we note that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^j}{\partial\eta^m}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial\widehat{x}^j}{\partial\eta^m}\right) + \frac{\partial}{\partial\eta^n}\left(\frac{\partial\widehat{x}^j}{\partial\eta^m}\right)\frac{\mathrm{d}\widehat{\eta}^n}{\mathrm{d}t}\\
&= \frac{\partial\overline{U}^j}{\partial\eta^m} + \frac{\partial^2\widehat{x}^j}{\partial\eta^n\partial\eta^m}\frac{\mathrm{d}\widehat{\eta}^n}{\mathrm{d}t}\\
&= \frac{\partial}{\partial\eta^m}\left(\frac{\partial\widehat{x}^j}{\partial\eta^n}U^n\right) + \frac{\partial^2\widehat{x}^j}{\partial\eta^n\partial\eta^m}\frac{\mathrm{d}\widehat{\eta}^n}{\mathrm{d}t}\\
&= \frac{\partial^2\widehat{x}^j}{\partial\eta^m\partial\eta^n}U^n + \frac{\partial\widehat{x}^j}{\partial\eta^n}\frac{\partial U^n}{\partial\eta^m} + \frac{\partial^2\widehat{x}^j}{\partial\eta^n\partial\eta^m}\frac{\mathrm{d}\widehat{\eta}^n}{\mathrm{d}t},
\end{aligned}$$

where $\overline{U}^j$ are Cartesian velocity components of the coordinate flow. A multiplication of the latter expression by $\partial\widehat{\eta}^i/\partial x^j$ gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^j}{\partial\eta^m}\right)\frac{\partial\widehat{\eta}^i}{\partial x^j} &= \underbrace{\frac{\partial\widehat{\eta}^i}{\partial x^j}\frac{\partial^2\widehat{x}^j}{\partial\eta^m\partial\eta^n}}_{\Gamma^i_{nm}}U^n + \underbrace{\frac{\partial\widehat{\eta}^i}{\partial x^j}\frac{\partial\widehat{x}^j}{\partial\eta^n}}_{\delta^i_n}\frac{\partial U^n}{\partial\eta^m} + \underbrace{\frac{\partial\widehat{\eta}^i}{\partial x^j}\frac{\partial^2\widehat{x}^j}{\partial\eta^n\partial\eta^m}}_{\Gamma^i_{nm}}\frac{\mathrm{d}\widehat{\eta}^n}{\mathrm{d}t}\\
&= \frac{\partial U^i}{\partial\eta^m} + \Gamma^i_{nm}U^n + \Gamma^i_{nm}\frac{\mathrm{d}\widehat{\eta}^n}{\mathrm{d}t}\\
&= U^i_{;m} + \Gamma^i_{nm}\frac{\mathrm{d}\widehat{\eta}^n}{\mathrm{d}t}.
\end{aligned} \tag{3.7}$$

Now we substitute (3.7) into (3.6)

$$\frac{\delta A^i_k}{\delta t} = \frac{\partial A^i_k}{\partial t} + \frac{\partial A^i_k}{\partial \eta_m}\frac{\mathrm{d}\widehat{\eta}^m}{\mathrm{d}t} + \left(U^i_{;m} + \Gamma^i_{nm}\frac{\mathrm{d}\widehat{\eta}^n}{\mathrm{d}t}\right)A^m_k - \left(U^m_{;k} + \Gamma^m_{kn}\frac{\mathrm{d}\widehat{\eta}^n}{\mathrm{d}t}\right)A^i_m$$

and collect all terms multiplying $\mathrm{d}\widehat{\eta}^j/\mathrm{d}t = u^j - U^j$. We obtain the following final expression:

$$\boxed{\frac{\delta A^i_k}{\delta t} = \frac{\partial A^i_k}{\partial t} + A^i_{k;m}(u^m - U^m) + A^m_k U^i_{;m} - A^i_m U^m_{;k}.} \tag{3.8}$$

It is easy to repeat the proof for a fully covariant or contravariant second order tensor

$$\frac{\delta A_{ik}}{\delta t} = \frac{\partial A_{ik}}{\partial t} + A_{ik;m}(u^m - U^m) - A_{im}U^m_{;k} - A_{mk}U^m_{;i}, \tag{3.9}$$

$$\frac{\delta A^{ik}}{\delta t} = \frac{\partial A^{ik}}{\partial t} + A^{ik}_{;m}(u^m - U^m) + A^{im}U^k_{;m} + A^{mk}U^i_{;m}. \tag{3.10}$$

It is also easy to determine the following expressions for the intrinsic derivative of scalars, contravariant (Luo and Bewley (2004)) and covariant vector fields:

$$\frac{\delta \mathfrak{F}}{\delta t} = \frac{\partial \mathfrak{F}}{\partial t} + \mathfrak{F}_{;k}(u^k - U^k), \tag{3.11}$$

$$\frac{\delta A^i}{\delta t} = \frac{\partial A^i}{\partial t} + A^i_{;k}(u^k - U^k) + A^k U^i_{;k}, \tag{3.12}$$

$$\frac{\delta A_i}{\delta t} = \frac{\partial A_i}{\partial t} + A_{i;k}(u^k - U^k) - A_k U^k_{;i}. \tag{3.13}$$

3.2. *Intrinsic derivative of the metric tensor*

It is well known that the covariant derivative of the metric tensor vanishes identically (e.g., Weinberg (1972), Aris (1989)), i.e. $g_{ij;m} = 0$. Therefore, according to (3.9), the intrinsic derivative of g_{ij} is

$$\frac{\delta g_{ij}}{\delta t} = \frac{\partial g_{ij}}{\partial t} - g_{im}U^m_{;j} - g_{mj}U^m_{;i}. \tag{3.14}$$

If the coordinate system is fixed (i.e., $U^i = 0$ and g_{im} is not a function of t) we obtain $\delta g_{ik}/\delta t = 0$. The covariance of the expression (3.14) guarantees that the same conclusion *persists* also if time-dependent curvilinear coordinates are considered. We verify this assertion by a direct calculation. As easily seen

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial \widehat{x}^p}{\partial \eta^i}\frac{\partial \widehat{x}^p}{\partial \eta^j}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial \widehat{x}^p}{\partial \eta^i}\right)\frac{\partial \widehat{x}^p}{\partial \eta^j} + \frac{\partial \widehat{x}^p}{\partial \eta^i}\frac{\partial}{\partial t}\left(\frac{\partial \widehat{x}^p}{\partial \eta^j}\right) \\
&= \frac{\partial \overline{U}^p}{\partial \eta^i}\frac{\partial \widehat{x}^p}{\partial \eta^j} + \frac{\partial \widehat{x}^p}{\partial \eta^i}\frac{\partial \overline{U}^p}{\partial \eta^j} \\
&= \frac{\partial U^m}{\partial \eta^i}\underbrace{\frac{\partial \widehat{x}^p}{\partial \eta^m}\frac{\partial \widehat{x}^p}{\partial \eta^j}}_{g_{mj}} + U^n\underbrace{\frac{\partial^2 \widehat{x}^p}{\partial \eta^n \partial \eta^i}\frac{\partial \widehat{x}^p}{\partial \eta^j}}_{[ni,j]} + \frac{\partial U^m}{\partial \eta^i}\underbrace{\frac{\partial \widehat{x}^p}{\partial \eta^m}\frac{\partial \widehat{x}^p}{\partial \eta^i}}_{g_{mi}} + U^n\underbrace{\frac{\partial^2 \widehat{x}^p}{\partial \eta^n \partial \eta^j}\frac{\partial \widehat{x}^p}{\partial \eta^i}}_{[nj,i]} \\
&= \frac{\partial U^m}{\partial \eta^i}g_{mj} + U^n[ni,j] + \frac{\partial U^m}{\partial \eta^j}g_{mi} + U^n[nj,i],
\end{aligned} \tag{3.15}$$

where $[mi, j]$ denotes the Christoffel symbol of the first kind and $\overline{U}^p$ are Cartesian velocity components of the coordinate flow. Now, using the well-known identity

$$[ni, j] = \Gamma^m_{ni} g_{mj}, \tag{3.16}$$

we re-write (3.15) as

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= g_{mj}\left(\frac{\partial U^m}{\partial \eta^i} + U^n\Gamma^m_{ni}\right) + g_{mi}\left(\frac{\partial U^m}{\partial \eta^j} + U^n\Gamma^m_{nj}\right) \\ &= g_{mj}U^m_{;i} + g_{mi}U^m_{;j}. \end{aligned} \tag{3.17}$$

This means that, independently from the status of motion of the curvilinear coordinate system, the metric tensor is always parallel transported[5], i.e.

$$\boxed{\frac{\delta g_{ij}}{\delta t} = 0.} \tag{3.18}$$

4. Convected differentiation

Convected time derivatives were established a long time ago by Oldroyd (1950) for the purpose of rigorously characterizing rheological properties of a moving continuum. These types of derivatives (here denoted by $\mathrm{d}_c/\mathrm{d}t$) basically state the general covariance of a material variation of a tensor along a fluid trajectory in convected coordinates. In its original formulation Oldroyd developed these concepts in *fixed* curvilinear systems (see also Scriven (1960), Aris (1989)). In this section we extend this classical theory to include the possibility of time-dependent coordinate flows. To this end, we consider a second order mixed tensor $\alpha^i_j(\boldsymbol{\xi}, t)$ in material coordinates and we set down the following transformation rule:

$$\frac{\partial\widehat{\eta}^m}{\partial\xi^i}\alpha^i_j = \frac{\partial\widehat{\eta}^n}{\partial\xi^j}A^m_n, \tag{4.1}$$

where $A^m_n(\boldsymbol{\eta}, t)$ are components of α^i_j in an arbitrary time-dependent curvilinear system. We differentiate both sides of (4.1) with respect to time keeping $\boldsymbol{\xi}$ constant (i.e. following the physical particle $\boldsymbol{\xi}$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{\eta}^m}{\partial\xi^i}\right)\alpha^i_j + \frac{\partial\widehat{\eta}^m}{\partial\xi^i}\frac{\mathrm{d}\alpha^i_j}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{\eta}^n}{\partial\xi^j}\right)A^m_n + \frac{\partial\widehat{\eta}^n}{\partial\xi^j}\frac{\mathrm{d}A^m_n}{\mathrm{d}t}. \tag{4.2}$$

From (2.1), it is easy to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{\eta}^m}{\partial\xi_i}\right) = \left(-\frac{\partial U^m}{\partial\eta^q} + \frac{\partial u^m}{\partial\eta^q}\right)\frac{\partial\eta^q}{\partial\xi^i}, \tag{4.3}$$

recalling that $\mathrm{d}/\mathrm{d}t$ denotes temporal differentiation at $\boldsymbol{\xi}$ constant. An alternative derivation of equation (4.3) is presented in appendix B. A substitution of (4.3) into (4.2) yields

$$\left(-\frac{\partial U^m}{\partial\eta^q} + \frac{\partial u^m}{\partial\eta^q}\right)\frac{\partial\eta^q}{\partial\xi^i}\alpha^i_j + \frac{\partial\widehat{\eta}^m}{\partial\xi^i}\frac{\mathrm{d}\alpha^i_j}{\mathrm{d}t} = \left(-\frac{\partial U^n}{\partial\eta^q} + \frac{\partial u^n}{\partial\eta^q}\right)\frac{\partial\eta^q}{\partial\xi^j}A^m_n + \frac{\partial\widehat{\eta}^n}{\partial\xi^j}\frac{\mathrm{d}A^m_n}{\mathrm{d}t}. \tag{4.4}$$

Multiplying (4.4) by $\partial\widehat{\xi}^j/\partial\eta^r$ and rearranging all terms give

$$\frac{\mathrm{d}_c A^m_r}{\mathrm{d}t} := \frac{\partial A^m_r}{\partial t} + \frac{\partial A^m_r}{\partial\eta^j}(u^j - U^j) - A^j_r\left(\frac{\partial u^m}{\partial\eta^j} - \frac{\partial U^m}{\partial\eta^j}\right) + A^m_j\left(\frac{\partial u^j}{\partial\eta^r} - \frac{\partial U^j}{\partial\eta^r}\right). \tag{4.5}$$

[5] For a rigorous definition of parallel transport of a vector or a tensor along a particle path see Aris (1989) or Weinberg (1972).

Equation (4.5) is the generalization of the convected derivative introduced by Oldroyd (1950) in the context of fixed curvilinear coordinates. It is easy to check that (4.4) is covariant with respect to time-dependent transformations. To see this we simply substitute

$$\frac{\partial A_r^m}{\partial \eta^j} = A^m_{r;j} - \Gamma^m_{jp} A^p_r + \Gamma^p_{rj} A^m_p, \tag{4.6}$$

$$\frac{\partial u^m}{\partial \eta^j} = u^m_{;j} - \Gamma^m_{pj} u^p, \tag{4.7}$$

$$\frac{\partial U^m}{\partial \eta^j} = U^m_{;j} - \Gamma^m_{pj} U^p, \tag{4.8}$$

into (4.5) to obtain the final result

$$\boxed{\frac{\mathrm{d}_c A_r^m}{\mathrm{d}t} = \frac{\partial A_r^m}{\partial t} + A^m_{r;j}(u^j - U^j) + A^m_j (u^j - U^j)_{;r} - A^j_r (u^m - U^m)_{;j}.} \tag{4.9}$$

If the curvilinear coordinate system is fixed, i.e. $U^m = 0$, we obtain the classical Oldroyd derivative. Note also that if the curvilinear coordinate system is in motion with the fluid, i.e. $U^m = u^m$, then (4.9) degenerates in a simple partial derivative with respect to time. It is easy to obtain a fully covariant expression for other types of tensors if we add one term like $A^m_j (u^j - U^j)_{;r}$ for every covariant index and one like $-A^j_r (u^m - U^m)_{;j}$ for every contravariant index. The application of this simple rule leads to the following convected derivatives:

$$\frac{\mathrm{d}_c A_{mr}}{\mathrm{d}t} = \frac{\partial A_{mr}}{\partial t} + A_{mr;j}(u^j - U^j) + A_{mj}(u^j - U^j)_{;r} + A_{jr}(u^j - U^j)_{;m}, \tag{4.10}$$

$$\frac{\mathrm{d}_c A^{mr}}{\mathrm{d}t} = \frac{\partial A^{mr}}{\partial t} + A^{mr}_{;j}(u^j - U^j) - A^{mj}(u^r - U^r)_{;j} - A^{jr}(u^m - U^m)_{;j}. \tag{4.11}$$

4.1. Connections between intrinsic and convected derivatives

A straightforward comparison between the equations obtained in sections 3 and 4 allows us to establish general relations between the convected and the intrinsic derivatives of scalars, vectors and tensors. For instance,

$$\frac{\mathrm{d}_c \mathfrak{F}}{\mathrm{d}t} = \frac{\delta \mathfrak{F}}{\delta t}. \tag{4.12}$$

Analogously for vector and tensor fields

$$\frac{\mathrm{d}_c B^i}{\mathrm{d}t} = \frac{\delta B^i}{\delta t} - B^m u^i_{;m}, \tag{4.13}$$

$$\frac{\mathrm{d}_c B_i}{\mathrm{d}t} = \frac{\delta B_i}{\delta t} + B_m u^m_{;i}, \tag{4.14}$$

$$\frac{\mathrm{d}_c A^{ij}}{\mathrm{d}t} = \frac{\delta A^{ij}}{\delta t} - A^{im} u^j_{;m} - A^{mj} u^i_{;m}, \tag{4.15}$$

$$\frac{\mathrm{d}_c A_{ij}}{\mathrm{d}t} = \frac{\delta A_{ij}}{\delta t} + A_{im} u^m_{;j} + A_{mj} u^m_{;i}. \tag{4.16}$$

A simple application of (4.16) shows that the convected derivative of the metric tensor is

$$\frac{\mathrm{d}_c g_{ij}}{\mathrm{d}t} = g_{im} u^m_{;j} + g_{mj} u^m_{;i}. \tag{4.17}$$

In fact we have seen in section 3.2 that the intrinsic derivative of g_{ij} vanishes identically. This basically shows that the generalized convected derivative is not *compatible*[6]. The quantity on the right-hand side of (4.17) is twice the velocity deformation tensor

$$e_{ij} = \tfrac{1}{2}(u_{i;j} + u_{j;i}). \tag{4.18}$$

Note that the convected derivative of the metric tensor does not depend on the velocity of the coordinate flow, since it represents the rate of strain of a fluid element which is obviously independent from the status of motion of the coordinate system. Rheological properties of materials which are functions of this local rate of strain satisfy the principle of material indifference, i.e. the response of the material is the same for all observers. Notable examples are the Stokesian fluid and the linear and isotropic Newtonian fluid, discussed in section 7.1.

5. The acceleration addition theorem

As a first application of the theoretical apparatus developed in the previous sections we consider the acceleration addition theorem for a fluid flow as described from time-dependent curvilinear coordinates. This is equivalent to considering the following question: if we *observe* a physical flow from another flow of coordinates what is the expression of the inertial forces? The answer has been well known for centuries if rectangular reference frames in rigid motion are considered. Here we obtain an elegant generalization of this classical result by using the intrinsic derivative concept developed in section 3. To this end we recall that for an arbitrary contravariant vector A^i:

$$\frac{\partial A^k}{\partial t} + A^k_{;j} u^j_R = \frac{\delta A^k}{\delta t} - A^p U^k_{;p}. \tag{5.1}$$

By examining (2.4) under this new perspective it is easy to obtain $\left(u^j_R = u^j - U^j\right)$:

$$\frac{\mathrm{d}^2 \widehat{\eta}^k}{\mathrm{d}t^2} = \frac{\delta u^k}{\delta t} - \frac{\delta U^k}{\delta t} - u^p_R U^k_{;p} - \Gamma^k_{nj} u^j_R u^n_R, \tag{5.2}$$

which is equivalent to

$$\boxed{\frac{\delta u^k}{\delta t} = \frac{\delta U^k}{\delta t} + \frac{\mathrm{d}u^k_R}{\mathrm{d}t} + \Gamma^k_{nj} u^j_R u^n_R + u^p_R U^k_{;p}.} \tag{5.3}$$

Equation (5.3) represents the acceleration addition theorem in its fully *covariant* form[7]. It is easy to show that this result includes the classical acceleration addition theorem for rectangular reference frames in rigid motion (Batchelor (1967), section 3) as a subcase. Moreover, as easily seen from (5.3), the relative acceleration $\mathrm{d}u^k_R/\mathrm{d}t$ in general is not a tensor, due to the presence of $\Gamma^k_{nj} u^j_R u^n_R$. The difference between $\mathrm{d}u^k_R/\mathrm{d}t$ and $\delta u^k_R/\delta t$ has already been clearly pointed out in the introduction of section 3.

[6] As pointed out by Thiffeault (2001), a compatible operator vanishes when acting on the metric.

[7] Note that an intrinsic differentiation of $u^i = U^i + u^i_R$ immediately leads to

$$\frac{\delta u^k}{\delta t} = \frac{\delta U^k}{\delta t} + \frac{\delta u^k_R}{\delta t}, \tag{5.4}$$

which coincides with (5.3).

6. The Reynolds transport theorem

We consider a fluid flow described from an arbitrary time-dependent flow of coordinates. We are interested in the quantity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \mathfrak{F}\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3, \tag{6.1}$$

where $\mathfrak{F}(\boldsymbol{\eta}, t)$ denotes a scalar field, $V(t)$ is a material volume convected by the fluid and $g(\boldsymbol{\eta}, t)$ is the time-dependent determinant of the metric tensor. Expression (6.1) can be easily obtained from standard calculus by considering a change of variables from Cartesian to time-dependent curvilinear coordinates and noting that the modulus of the Jacobian determinant for this transformation is exactly $\sqrt{g}$. In the very special case where the coordinate flow and the fluid flow coincide, the domain of the integral (6.1) *loses* its time dependence and this allows us to write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \mathfrak{F}\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3 &= \int_{V_0}\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{F}'\sqrt{\gamma})\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi^3\\ &= \int_{V_0}\left(\frac{\mathrm{d}\mathfrak{F}'}{\mathrm{d}t}\sqrt{\gamma}+\mathfrak{F}'\frac{1}{2\sqrt{\gamma}}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)\mathrm{d}\xi^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi^3,\end{aligned} \tag{6.2}$$

where $\gamma(\boldsymbol{\xi}, t)$ denotes the time-dependent metric tensor determinant in material (convected) coordinates. The analytical expression of $\mathfrak{F}$ changes as well when we perform the coordinate transformation[8]. In appendix A we obtain that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma\gamma^{ij}\frac{\mathrm{d}\gamma_{ij}}{\mathrm{d}t}. \tag{6.3}$$

A substitution of (6.3) into (6.2) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \mathfrak{F}\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3 = \int_{V_0}\left(\frac{\mathrm{d}\mathfrak{F}'}{\mathrm{d}t}+\mathfrak{F}'\frac{1}{2}\gamma^{ij}\frac{\mathrm{d}\gamma_{ij}}{\mathrm{d}t}\right)\sqrt{\gamma}\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi^3.$$

Now we need to transform the right-hand side of this expression back to the moving curvilinear system $\boldsymbol{\eta}$. The scalar field $\gamma^{ij}\mathrm{d}\gamma_{ij}/\mathrm{d}t$ transforms according to the product of two second order tensors: γ^{ij} and $\mathrm{d}\gamma_{ij}/\mathrm{d}t$. To obtain the expression of this scalar in coordinates $\boldsymbol{\eta}$ we first transform singularly both γ^{ij} and $\mathrm{d}\gamma_{ij}/\mathrm{d}t$, then we perform the direct product and contract the indices. The components of $\mathrm{d}\gamma_{ij}/\mathrm{d}t$ in coordinates $\boldsymbol{\eta}$ are generalized convected derivatives (see section 4), which we have denoted by $\mathrm{d}_c g_{ij}/\mathrm{d}t$. According to (4.17) we have the Euler formula

$$\begin{aligned}\frac{1}{2}g^{ij}\frac{\mathrm{d}_c g_{ij}}{\mathrm{d}t} &= \frac{1}{2}g^{ij}\left(g_{im}u^m_{;j}+g_{mj}u^m_{;i}\right)\\ &= u^m_{;m}\end{aligned} \tag{6.4}$$

and therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \mathfrak{F}\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3 &= \int_{V(t)}\left(\frac{\mathrm{d}_c\mathfrak{F}}{\mathrm{d}t}+\frac{1}{2}\mathfrak{F}g^{ij}\frac{\mathrm{d}_c g_{ij}}{\mathrm{d}t}\right)\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3\\ &= \int_{V(t)}\left(\frac{\delta\mathfrak{F}}{\delta t}+\mathfrak{F}u^m_{;m}\right)\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3,\end{aligned} \tag{6.5}$$

where we have also used the identity (4.12). This completes the proof of the Reynolds transport theorem in curvilinear time-dependent coordinates. The final formula (6.5) has been used by Luo and Bewley (2004) to obtain the contravariant form of the Navier–Stokes equations in curvilinear time-dependent coordinates.

[8] As an example consider $\mathfrak{F}(x, y) = x^2+y^2$ in Cartesian coordinates (x, y). When we transform to polar coordinates, $\mathfrak{F}(x, y)$ becomes $\mathfrak{F}'(r, \theta) = r^2$.

7. Mass conservation and momentum transport

We consider the fluid mass density $\rho(\boldsymbol{\eta}, t)$ as expressed in curvilinear time-dependent coordinates. The total mass of a material volume convected by the fluid is a constant and therefore by applying the Reynolds transport theorem we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3 = \int_{V(t)} \left(\frac{\delta\rho}{\delta t} + \rho u^m_{;m}\right)\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3 = 0. \tag{7.1}$$

This is true for an arbitrary $V(t)$ and hence (7.1) leads to

$$\frac{\delta\rho}{\delta t} + \rho u^m_{;m} = 0, \tag{7.2}$$

which can be equivalently written as

$$\frac{\partial\rho}{\partial t} + (\rho u^m)_{;m} - \frac{\partial\rho}{\partial\eta^k}U^k = 0. \tag{7.3}$$

This is the equation of *continuity* expressed in time-dependent curvilinear coordinates. Now we consider the scalar field obtained by a direct product of the contravariant momentum density ρu^i and a parallel covariant vector field l_i, i.e. we consider the components of the momentum density projected along l_i. A trivial application of the transport theorem in this case leads to the following momentum transport equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho u^i l_i\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3 = \int_{V(t)} \rho\frac{\delta u^i}{\delta t} l_i\sqrt{g}\,\mathrm{d}\eta^1\,\mathrm{d}\eta^2\,\mathrm{d}\eta^3. \tag{7.4}$$

Using similar arguments one can easily reformulate the whole dynamical theory for fluid flows and heat transfer in time-dependent coordinate systems from an integral point of view. For instance it is easy to prove that the momentum conservation leads to the following Cauchy's equation of motion:

$$\rho\frac{\delta u^i}{\delta t} = \rho f^i + T^{ij}_{;j}, \tag{7.5}$$

where T^{ij} is the stress tensor and f^i is the body force vector. Analogously, the intrinsic form of the first law of thermodynamics applied to a fluid element in arbitrary motion leads to the following Fourier equation:

$$\rho c_p\frac{\delta T}{\delta t} = \mathcal{D} - q^m_{;m} + q_g + \beta T\frac{\delta p}{\delta t}, \tag{7.6}$$

where T denotes the temperature, ρ is the density, c_p is the specific heat at constant pressure, $\mathcal{D}$ is the viscous dissipation, q^m are the contravariant components of the heat flux, q_g is a source term, p is the thermodynamic pressure and β is the isobaric compressibility coefficient.

7.1. Newtonian fluids

The stress tensor for a linear and isotropic Newtonian fluid in convected coordinates is (e.g. Aris (1989), p 189)

$$\begin{aligned}\Theta^{ij} &= -p\gamma^{ij} + \lambda\gamma^{ij}\gamma^{pq}\varepsilon_{pq} + \mu(\gamma^{ip}\gamma^{jq} + \gamma^{iq}\gamma^{jp})\varepsilon_{pq}\\ &= -p\gamma^{ij} + \frac{1}{2}\lambda\gamma^{ij}\gamma^{pq}\frac{\mathrm{d}\gamma_{pq}}{\mathrm{d}t} + \mu\gamma^{ip}\gamma^{jq}\frac{\mathrm{d}\gamma_{pq}}{\mathrm{d}t}.\end{aligned} \tag{7.7}$$

Transforming this back to curvilinear time-dependent coordinates we obtain

$$T^{ij} = -pg^{ij} + \frac{1}{2}\lambda g^{ij} \underbrace{g^{pq}\frac{\mathrm{d}_c g_{pq}}{\mathrm{d}t}}_{2u^m_{;m}} + \mu g^{ip}g^{jq}\frac{\mathrm{d}_c g_{pq}}{\mathrm{d}t}$$
$$= -pg^{ij} + \lambda g^{ij}u^m_{;m} + \mu g^{ip}g^{jq}(u_{p;q} + u_{q;p}). \tag{7.8}$$

As easily seen the stress tensor components do *not* depend on the velocity of the coordinate system $\boldsymbol{\eta}$.[9] The divergence of T^{ij} is easily computed as

$$T^{ij}_{;j} = \left(-p_{;j} + \lambda u^m_{;mj}\right)g^{ij} + \mu g^{ip}g^{jq}(u_{p;qj} + u_{q;pj})$$
$$= \left(-p_{;j} + (\lambda+\mu)u^m_{;mj}\right)g^{ij} + \mu g^{jq}u^i_{;jq}. \tag{7.9}$$

A substitution of this expression into (7.5) leads to the contravariant form of the Navier–Stokes equations in time-dependent curvilinear coordinates (e.g. Luo and Bewley (2004)).

8. Summary

We have established a fully covariant formulation of kinematics and dynamics of fluid flows in time-dependent curvilinear coordinate systems. These moving and deformable reference frames have the same properties of a fluid motion and can be successfully used both for numerical and theoretical investigations. We have extended the Reynolds transport theorem, the Euler formula as well as the acceleration addition theorem to these general types of coordinates by generalizing the convected differentiation concept originally introduced by Oldroyd (1950). In this context, previously unobserved connections between convected and intrinsic derivatives are reported. A rigorous formulation of dynamical equations and conservation laws in curvilinear time-dependent coordinates could be the key for the construction of variational principles based on the method of constrained variations. In fact, classical existence conditions for a potential of a nonlinear operator (e.g. Vainberg (1964), Filippov (1989), Finlayson (1972), Tonti (1984), Tonti (1969a), Tonti (1969b)) can be recast into the search for a symmetrizing coordinate flow whose physical properties need to be carefully investigated.

Appendix A. Material derivative of the metric tensor determinant in convected coordinates

Consider a curvilinear coordinate system which is convected by a fluid flow. Let γ_{ij} be the metric tensor associated with this coordinate system. If we denote by Γ^{ij} the co-factor of the matrix element γ_{ij}, we can easily write the determinant of the metric tensor γ_{ij} as

$$\gamma := \det(\gamma_{ij}) = \sum_j \gamma_{ij}\Gamma^{ij} \quad \forall i, \qquad \text{(no sum over } i). \tag{A.1}$$

If we also sum over i we find

$$3\gamma = \gamma_{ij}\Gamma^{ij} \qquad \text{(tridimensional case)}, \tag{A.2}$$

$$2\gamma = \gamma_{ij}\Gamma^{ij} \qquad \text{(bidimensional case)}. \tag{A.3}$$

For the tridimensional case the co-factor of the element γ_{ij} has tensorial expansion

$$\Gamma^{ij} = \tfrac{1}{2}\epsilon^{imn}\epsilon^{jpq}\gamma_{mp}\gamma_{nq}, \tag{A.4}$$

[9] This is known as the principle of material indifference: the response of the material is the same for all observers.

where ϵ^{imn} is the permutation symbol, i.e. the Levi–Civita tensorial density. A substitution of (A.4) into (A.2) leads to

$$\gamma = \tfrac{1}{6}\epsilon^{imn}\epsilon^{jpq}\gamma_{ij}\gamma_{mp}\gamma_{nq}. \tag{A.5}$$

By taking the material derivative of (A.5) we find

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{6}\epsilon^{imn}\epsilon^{jpq}\left(\frac{\mathrm{d}\gamma_{ij}}{\mathrm{d}t}\gamma_{mp}\gamma_{nq} + \gamma_{ij}\frac{\mathrm{d}\gamma_{mp}}{\mathrm{d}t}\gamma_{nq} + \gamma_{ij}\gamma_{mp}\frac{\mathrm{d}\gamma_{nq}}{\mathrm{d}t}\right). \tag{A.6}$$

An interchange of the summation indices in the second term ($m \leftrightarrow i$, $p \leftrightarrow j$) as well as in the third term ($n \leftrightarrow i$, $q \leftrightarrow j$) yields[10]

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{6}\epsilon^{imn}\epsilon^{jpq}3\frac{\mathrm{d}\gamma_{ij}}{\mathrm{d}t}\gamma_{mp}\gamma_{nq} = \underbrace{\frac{1}{2}\epsilon^{imn}\epsilon^{jpq}\gamma_{mp}\gamma_{nq}}_{\Gamma^{ij}}\frac{\mathrm{d}\gamma_{ij}}{\mathrm{d}t}. \tag{A.7}$$

In the sense of matrices, the contravariant form of the metric tensor γ^{ij} is the inverse of γ_{ij}, i.e. $\gamma_{ik}\gamma^{kj} = \delta_i^j$. From the inversion theory of square matrices it follows that

$$\gamma^{ij} = \frac{\Gamma^{ij}}{\gamma} \quad \Rightarrow \quad \Gamma^{ij} = \gamma\gamma^{ij}. \tag{A.8}$$

Substituting this expression back to (A.7) yields the elegant formula

$$\boxed{\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma\gamma^{ij}\frac{\mathrm{d}\gamma_{ij}}{\mathrm{d}t}.} \tag{A.9}$$

We remark that the quantity $\gamma^{ij}\mathrm{d}\gamma_{ij}/\mathrm{d}t$ is a scalar because the material derivative of any tensor in convected coordinates is a tensor (see section 4).

Appendix B. An alternative derivation of equation (4.3)

In this section we provide an alternative derivation of equation (4.3). To this end we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{\eta}^m}{\partial\xi^i}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{\eta}^m}{\partial x^l}\right)\frac{\partial\widehat{x}^l}{\partial\xi^i} + \frac{\partial\widehat{\eta}^m}{\partial x^l}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^l}{\partial\xi^i}\right). \tag{B.1}$$

The first term on the right-hand side is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{\eta}^m}{\partial x^l}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial\widehat{\eta}^m}{\partial x^l}\right) + \frac{\partial}{\partial x^q}\left(\frac{\partial\widehat{\eta}^m}{\partial x^l}\right)\\ &= -\frac{\partial U^m}{\partial x^l} + \frac{\partial^2\widehat{\eta}^m}{\partial x^l\partial x^q}\frac{\mathrm{d}\widehat{x}^q}{\mathrm{d}t}\\ &= -\frac{\partial U^m}{\partial\eta^q}\frac{\partial\widehat{\eta}^q}{\partial x^l} + \frac{\partial^2\widehat{\eta}^m}{\partial x^l\partial x^q}\overline{u}^q\\ &= -\frac{\partial U^m}{\partial\eta^q}\frac{\partial\widehat{\eta}^q}{\partial x^l} + \frac{\partial^2\widehat{\eta}^m}{\partial x^l\partial x^q}\frac{\partial\widehat{x}^q}{\partial\eta^n}u^n.\end{aligned} \tag{B.2}$$

The second term on the right-hand side of the latter expression has something to do with the affine connection. In fact if we differentiate the identity

$$\frac{\partial\widehat{\eta}^m}{\partial x^k}\frac{\partial\widehat{x}^k}{\partial\eta^q} = \delta_q^m \tag{B.3}$$

[10] The aforementioned interchanges generate sign inversions in ϵ^{imn} and ϵ^{jpq} which neutralize each other in the sense that $\epsilon^{imn}\epsilon^{jpq} = (-\epsilon^{min})(-\epsilon^{pjq}) = \epsilon^{min}\epsilon^{pjq}$.

with respect to x^j we obtain (e.g. Lovelock and Rund (1989) p 70, remark 5)

$$\frac{\partial^2\widehat{\eta}^m}{\partial x^k \partial x^j}\frac{\partial\widehat{x}^k}{\partial\eta^q}+\underbrace{\frac{\partial\widehat{\eta}^m}{\partial x^k}\frac{\partial^2\widehat{x}^k}{\partial\eta^q\partial\eta^l}}_{\Gamma^m_{ql}}\frac{\partial\widehat{\eta}^l}{\partial x^j}=0, \tag{B.4}$$

that is

$$\frac{\partial^2\widehat{\eta}^m}{\partial x^k \partial x^j}\frac{\partial\widehat{x}^k}{\partial\eta^q}=-\Gamma^m_{ql}\frac{\partial\widehat{\eta}^l}{\partial x^j}. \tag{B.5}$$

Substituting this result back to (B.2), multiplied by $\partial\widehat{x}^l/\partial\xi^i$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{\eta}^m}{\partial x^k}\right)\frac{\partial\widehat{x}^l}{\partial\xi^i}&=-\frac{\partial U^m}{\partial\eta^q}\frac{\partial\widehat{\eta}^q}{\partial x^l}\frac{\partial\widehat{x}^l}{\partial\xi^i}-\Gamma^m_{np}u^n\frac{\partial\widehat{\eta}^p}{\partial x^l}\frac{\partial\widehat{x}^l}{\partial\eta^r}\frac{\partial\widehat{\eta}^r}{\partial\xi^i}\\&=-\frac{\partial U^m}{\partial\eta^q}\frac{\partial\widehat{\eta}^q}{\partial\xi^i}-\Gamma^m_{nq}u^n\frac{\partial\widehat{\eta}^q}{\partial\xi^i}\\&=\left(-\frac{\partial U^m}{\partial\eta^q}-\Gamma^m_{nq}u^n\right)\frac{\partial\widehat{\eta}^q}{\partial\xi^i}.\end{aligned} \tag{B.6}$$

Now we consider the second term on the right-hand side of (B.1):

$$\begin{aligned}\frac{\partial\widehat{\eta}^m}{\partial x^l}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{x}^l}{\partial\xi^i}\right)&=\frac{\partial\widehat{\eta}^m}{\partial x^l}\frac{\partial\overline{u}^l}{\partial\eta^q}\frac{\partial\widehat{\eta}^q}{\partial\xi^i}\\&=\frac{\partial\widehat{\eta}^m}{\partial x^l}\frac{\partial}{\partial\eta^q}\left(\frac{\partial\widehat{x}^l}{\partial\eta^p}u^p\right)\frac{\partial\widehat{\eta}^q}{\partial\xi^i}\\&=\frac{\partial\widehat{\eta}^m}{\partial x^l}\frac{\partial^2\widehat{x}^l}{\partial\eta^q\partial\eta^p}\frac{\partial\widehat{\eta}^q}{\partial\xi^i}u^p+\frac{\partial\widehat{\eta}^m}{\partial x^l}\frac{\partial\widehat{x}^l}{\partial\eta^p}\frac{\partial u^p}{\partial\eta^p}\frac{\partial\widehat{\eta}^q}{\partial\xi^i}\\&=\Gamma^m_{qp}u^p\frac{\partial\eta^q}{\partial\xi^i}+\frac{\partial u^m}{\partial\eta^q}\frac{\partial\widehat{\eta}^q}{\partial\xi^i}\\&=\left(\Gamma^m_{qp}u^p+\frac{\partial u^m}{\partial\eta^q}\right)\frac{\partial\eta^q}{\partial\xi^i}.\end{aligned} \tag{B.7}$$

Using (B.6) and (B.7) we can write (B.1) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widehat{\eta}^m}{\partial\xi_i}\right)=\left(-\frac{\partial U^m}{\partial\eta^q}+\frac{\partial u^m}{\partial\eta^q}\right)\frac{\partial\eta^q}{\partial\xi^i}, \tag{B.8}$$

which coincides with (4.3).

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